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Guided surface waves near cutoff

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The joint effects of weak nonlinearity and weak linear damping on the dominant, antisymmetric gravity wave (excited by torsional oscillations of a plane wavemaker about a vertical axis) near its cutoff frequency in a rectangular channel are investigated, following Barnard, Mahony & Pritchard (1977). The evolution equations for the envelope of this mode are derived from the variational formulation previously developed for the parametrically excited cross-wave problem (Miles & Becker 1988). They are equivalent to those of Barnard *et al.*, after correcting their damping and self-interaction terms, and, after appropriate normalization, differ from the cross-wave evolution equations only in the boundary condition at the wavemaker. Analytical approximations and the results of numerical integration for stationary envelopes (as observed in the experiments of Barnard *et al.*) are presented. The present results are somewhat closer to the observations of Barnard *et al.* than are their calculations, but the differences between the two calculations are not qualitatively significant.

1. Introduction

Following Barnard, Mahony & Pritchard (1977) and Kit, Miloh & Shemer (1987), hereinafter referred to as BMP and KMS, I consider antisymmetric (with respect to the vertical mid-plane) gravity waves of free-surface displacement ζ in a semi-infinite rectangular tank of width b and depth d driven by torsional oscillations of a plane wavemaker, for which the longitudinal displacement is given by

$$x = \chi(y, t) = -(y-\tfrac{1}{2}b)\tan(\epsilon \sin\omega t) \quad (0 < y < b, \quad -d < z < \zeta), \tag{1.1}$$

on the assumptions that: (i) $0 < \epsilon \ll 1$, (ii) ω approximates the cutoff frequency

$$\omega_1 = (gkT)^{\frac{1}{2}} \quad (k \equiv \pi/b,\ T \equiv \tanh kd), \tag{1.2}$$

and (iii) the flow is inviscid and irrotational (although dissipation is introduced subsequently as a small perturbation). The boundary condition (1.1) may be expanded in the complete set $\cos nky$ in $0 < y < b$ and in a complete set in $-d < z < 0$ that comprises the non-oscillatory term $\cosh k_0(z+d)$ plus an infinite, discrete set of oscillatory terms (cf. Havelock 1929), where k_0 is determined by

$$k_0 \tanh k_0 d = \omega^2/g \equiv \kappa, \tag{1.3}$$

(note that $k_0 = k$ for $\omega = \omega_1$). The linear approximation to the velocity potential then is

$$\phi = \frac{\epsilon\omega}{k}\left[Cf(x, t)\cos ky\frac{\cosh k_0(z+d)}{\cosh k_0 d} + \frac{\phi_0}{\kappa}\right], \tag{1.4}$$

wherein the first term represents the dominant (antisymmetric) mode,

$$f(x,t)=\begin{cases}(k_0^2-k^2)^{-\frac{1}{2}}\sin[(k_0^2-k^2)^{\frac{1}{2}}x-\omega t] & (k<k_0),\\ -(k^2-k_0^2)^{-\frac{1}{2}}\exp[-(k^2-k_0^2)^{\frac{1}{2}}x]\cos\omega t & (k>k_0),\end{cases} \tag{1.5}$$

$$C=\frac{8}{\pi}\left[\frac{T}{T+kd(1-T^2)}\right], \tag{1.6}$$

is derived from the boundary condition $\phi_x=\chi_t$ at $x=0$ (note that $f_x=\cos\omega t$ at $x=0$) through the aforementioned Fourier expansion, and the dimensionless potential ϕ_0 comprises the remaining modes, which decay exponentially away from the wavemaker.

It follows from (1.5) that the linear approximation fails, and either nonlinearity or dissipation or both must be significant, in the spectral neighbourhood of $k=k_0$. Weak dissipation may be incorporated through an extension of the linear analysis, which yields

$$f(x,t)=\mathscr{R}\{\lambda^{-1}\,e^{\lambda x-i\omega t}\},\quad \lambda=i(k_0^2-k^2+\tfrac{1}{2}i\pi Ck^2\delta)^{\frac{1}{2}}\quad(\mathscr{R}\lambda<0), \tag{1.7a, b}$$

in place of (1.5); $\mathscr{R}$ signifies the real part of, and δ is the ratio of actual to critical damping for the dominant mode. This damping ratio, which may be determined experimentally or estimated theoretically (Miles 1967), is typically small, and nonlinearity is at least as important as dissipation if $\delta=O(\epsilon)$, in which case dissipation may be neglected in the preliminary calculation of nonlinear effects and then introduced as a linear effect. Nonlinearity is negligible if $\epsilon\ll\delta=O(1)$.

I proceed as in the problem of parametrically excited cross-waves (Miles 1988, hereinafter I; Miles & Becker 1988, hereinafter II), starting from an extension of Luke's (1967) variational formulation. I then introduce appropriate trial functions for ϕ and ζ, average the resulting Lagrangian over the period $2\pi/\omega$, and invoke Hamilton's principle to obtain a nonlinear Schrödinger equation that governs the temporal and spatial modulation of the dominant mode. Finally, I incorporate weak damping and obtain analytical approximations and numerical solutions for stationary envelopes. My evolution equation differs from that of BMP in the incorporation of temporal modulation and in consequence of what appear to be errors in the sign of their damping term and in their self-interaction coefficient for the dominant mode (see Appendix). It is equivalent to that of KMS for deep-water ($kd\gg 1$) waves.

The question of whether my stationary solutions are stable, especially when there is more than one such solution for a particular set of parameters, remains open. The experimental results of BMP and KMS and the numerical solutions of the initial-value problem by KMS and Aranha, Yue & Mei (1982) suggest that at least some stationary solutions are stable for sufficiently small amplitude or sufficiently large damping, but that they may bifurcate to limit cycles, in which a periodic sequence of solitary waves propagates away from the wavemaker.

2. Variational formulation

The assumption of motion started from rest in an incompressible, inviscid fluid in the wave tank described in §1 leads to the boundary-value problem

$$\nabla^2\phi=0\quad(\chi<x<\infty,\quad 0<y<b,\quad -d<z<\zeta), \tag{2.1}$$

$$\phi_z=\zeta_t+\nabla\phi\cdot\nabla\zeta,\quad \phi_t+\tfrac{1}{2}(\nabla\phi)^2+g\zeta=0\quad(z=\zeta), \tag{2.2a, b}$$

$$\phi_y = 0 \quad (y = 0,\, b), \quad \phi_z = 0 \quad (z = -d), \tag{2.3a, b}$$

$$\phi_x = \chi_t + \nabla\phi\cdot\nabla\chi \quad (x = \chi), \tag{2.4}$$

$$\phi_x \to 0 \quad (x \uparrow \infty), \tag{2.5}$$

for the velocity potential $\phi(x, y, z, t)$ and the free-surface displacement $\zeta(x, y, t)$, where x, y, z are Cartesian coordinates and the subscripts signify partial differentiation. The null condition (2.5) may be either replaced or accompanied by a radiation condition.

The boundary-value problem (2.1)–(2.5) may be deduced through Hamilton's principle from the Lagrangian (Luke 1967; I §2),

$$\hat{L} = -\textstyle\iiint[\phi_t + \frac{1}{2}(\nabla\phi)^2 + gz]\,\mathrm{d}V, \tag{2.6}$$

where the volume integral is over the semi-infinite domain bounded by the wavemaker ($x = \chi$), the free surface ($z = \zeta$) and the fixed boundaries ($y = 0$, b and $z = -d$). An equivalent form, which incorporates the boundary conditions (2.3) and is more convenient for computation, is (I §2)

$$L = \frac{1}{2}\int_0^b \mathrm{d}y\left\{\iint\phi\nabla^2\phi\,\mathrm{d}x\,\mathrm{d}z + \int_{x_0}^{\infty}[\phi(2\zeta_t - \phi_z + \nabla\phi\cdot\nabla\zeta) - g\zeta^2]_{z=\zeta}\,\mathrm{d}x \right.$$
$$\left. + \int_{-d}^{z_0}[\phi(\phi_x - \nabla\chi\cdot\nabla\phi - 2\chi_t) - gz^2\chi_z]_{x=\chi}\,\mathrm{d}z\right\}, \tag{2.7}$$

where $x_0(y, t)$ and $z_0(y, t)$ are the coordinates of the intersection of the wavemaker ($x = \chi$) and the free surface ($z = \zeta$).

3. Average Lagrangian

Proceeding as in I, but allowing for spatial modulation through the slow variable X, we pose the trial functions

$$(k\omega/g)\,\phi = \epsilon\phi_0(x, y, z, \theta) + \epsilon^{\frac{1}{3}}\phi_1(y, z, \theta; X, \tau) + \epsilon\phi_{11}(y, z, \theta; X, \tau), \tag{3.1a}$$

$$k\zeta = \epsilon\zeta_0(x, y, \theta) + \epsilon^{\frac{1}{3}}\zeta_1(y, \theta; X, \tau) + \epsilon\zeta_{11}(y, \theta; X, \tau), \tag{3.1b}$$

where

$$\theta \equiv \omega t, \quad X \equiv \epsilon^{\frac{1}{3}}\kappa x, \quad \tau \equiv \epsilon\omega t. \tag{3.2a, b, c}$$

(The allowance for temporal modulation through the slow variable τ, even though we are concerned primarily with the stationary envelopes, facilitates the subsequent incorporation of linear damping.) The dimensionless pair (ϕ_0, ζ_0), where ϕ_0 is defined as in (1.4), satisfies (2.1), the linear approximations to (2.2a, b), (2.3a, b),

$$\phi_{0x} = -\kappa\left[k(y - \tfrac{1}{2}b) + C\cos ky\,\frac{\cosh k_0(z+d)}{\cosh k_0 d}\right]\cos\theta \quad (x = 0), \tag{3.3}$$

(obtained by linearizing (2.4) and subtracting out the contribution to ϕ_x of the dominant mode in (1.4)), and either (2.5) or the corresponding radiation condition. The pair

$$\phi_1 = A_{1\theta}(\theta; X, \tau)\cos ky\,\frac{\cosh k_0(z+d)}{\cosh k_0 d}, \quad \zeta_1 = A_1(\theta; X, \tau)\cos ky, \tag{3.4a, b}$$

where

$$A_1 = p(X, \tau)\cos\theta + q(X, \tau)\sin\theta = \mathscr{R}\{(p + \mathrm{i}q)\,\mathrm{e}^{-\mathrm{i}\theta}\}, \tag{3.5}$$

and ϕ_1 is the counterpart of the dominant mode in (1.4), satisfies the linear approximations to (2.2*a*, *b*) and (2.3*a*, *b*). We remark that ϕ_0 and ϕ_1 are orthogonal over $0 < y < b$, $-d < z < 0$.† The pair

$$\phi_{11} = \tfrac{1}{8}A_1 A_{1\theta}\left[-\left(\frac{3T^2+1}{T}\right)+3\left(\frac{1-T^4}{T^3}\right)\cos 2ky\,\frac{\cosh 2k_0(z+d)}{\cosh 2k_0 d}\right], \tag{3.6a}$$

$$\zeta_{11} = -\left(\frac{1-T^2}{4T}\right)\langle A_1^2\rangle+\frac{1}{4}\left[\frac{(T^2-1)(T^2+3)}{T^3}\langle A_1^2\rangle+\left(\frac{3-T^2}{T^3}\right)A_1^2\right]\cos 2ky, \tag{3.6b}$$

($\langle\,\rangle$ signifies an average over θ), which describes the self-interaction of (ϕ_1, ζ_1), has been chosen such that (3.1) satisfies (2.2)–(2.4) through $O(\epsilon)$. This last constraint determines (ϕ_{11}, ζ_{11}) within the additive pair $(-\theta F, F)$, where F is an arbitrary function of X and τ that must vanish in the present case in consequence of the requirement that ϕ_{11X} be bounded as $\theta\uparrow\infty$. (In the case of a finite tank it would be necessary to choose $F = \frac{1}{4}(T^{-1}-T)\langle A_1^2\rangle$ in order to ensure conservation of mass, but then A_1 is independent of X, so that $\phi_{11X}\equiv 0$.)

Substituting (3.1) into (2.7), invoking (3.2)–(3.6), and averaging the result over θ, we obtain (after a lengthy but straightforward reduction, in which X is adopted in place of x as a variable of integration and $pp_{XX}+qq_{XX}$, which is derived from $\langle\phi\nabla^2\phi\rangle$, is integrated by parts)

$$\mathscr{L} \equiv \epsilon^{\frac{1}{2}}\kappa\frac{\langle L-L_0\rangle}{g(\epsilon/k)^2 b} = \frac{1}{4}\int_0^\infty [p_\tau q - pq_\tau+\beta(p^2+q^2)+\tfrac{1}{16}D(p^2+q^2)^2$$
$$-\tfrac{1}{4}E(p_X^2+q_X^2)]\,\mathrm{d}X-\frac{1}{\pi}T^2 q\,|_{X=0}, \tag{3.7}$$

where L_0 is the Lagrangian for $p = q = 0$, an error factor of $1+O(\epsilon)$ is implicit,

$$\beta = \frac{1}{2\epsilon}\left(1-\frac{kT}{\kappa}\right) = \frac{\omega^2-\omega_1^2}{2\epsilon\omega^2}, \tag{3.8}$$

$$D = \frac{6T^6-5T^4+16T^2-9}{8T^4}, \quad E = T[T+kd(1-T^2)]. \tag{3.9a, b}$$

We note that $D \gtrless 0$ for $kd \gtrless 1.022$, $E > 0$ for $kd > 0$, and $D \sim E \sim 1$ for $kd \gg 1$. Nonlinearity is significant only at higher (than second) order in the spectral neighbourhood of $D = 0$.

4. Schrödinger equation

Invoking $\delta\int\mathscr{L}\mathrm{d}\tau = 0$, and combining the resulting evolution equations for p and q by introducing $r \equiv p+\mathrm{i}q$, we obtain the cubic Schrödinger equation

$$\mathrm{i}r_\tau+\tfrac{1}{4}Er_{XX}+(\beta+\tfrac{1}{8}D|r|^2)\,r = 0, \tag{4.1}$$

and the boundary condition

$$r_X = \mathrm{i}C \quad (X = 0), \tag{4.2}$$

which must be supplemented by a radiation or null condition as $X\uparrow\infty$.

† BMP apparently overlook this orthogonality and remark that 'it is virtually impossible to assess how accurate such a partitioning [between the dominant and remaining modes] is likely to be.' The present analysis removes this uncertainty.

Weak damping may be incorporated in (4.1) by replacing ∂_τ by $\partial_\tau + \mathrm{i}\alpha$, where

$$\alpha = \delta/\epsilon, \tag{4.3}$$

and δ is the damping ratio introduced in §1 (this replacement follows from the linear equations, in which ∂_t is replaced by $\partial_t + \delta\omega$ if $\delta \ll 1$). Damped stationary solutions then are governed by (4.1) with $\mathrm{i}r_\tau$ replaced by $\mathrm{i}\alpha r$ and (4.2). Re-scaling according to

$$r = (8\gamma/|D|)^{\frac{1}{2}} \mathscr{A}(\xi), \quad X = \tfrac{1}{2}(E/\gamma)^{\frac{1}{2}}\xi, \tag{4.4a, b}$$

$$\beta + \mathrm{i}\alpha = \gamma \mathrm{e}^{\mathrm{i}\phi} \quad (0 < \phi < \pi), \tag{4.5}$$

where $\gamma \equiv (\alpha^2 + \beta^2)^{\frac{1}{2}}$, and introducing

$$\sigma = 2^{-\frac{5}{2}} C |DE|^{\frac{1}{2}} \gamma^{-1} \tag{4.6a}$$

$$= 2^{-\frac{5}{2}} C |DE|^{\frac{1}{2}} \epsilon \left[\left(\frac{\omega^2 - \omega_1^2}{2\omega^2}\right)^2 + \delta^2\right]^{-\frac{1}{2}}, \tag{4.6b}$$

($2^{-\frac{5}{2}}C|DE|^{\frac{1}{2}} = 0.45$ for $kd \gg 1$), we obtain

$$\mathscr{A}_{\xi\xi} + (\mathrm{e}^{\mathrm{i}\phi} + |\mathscr{A}|^2 \operatorname{sgn} D)\mathscr{A} = 0, \tag{4.7}$$

and

$$\mathscr{A}_\xi = \mathrm{i}\sigma \quad (\xi = 0). \tag{4.8}$$

The null condition at $X = \infty$ may be replaced by

$$\mathscr{A}_\xi/\mathscr{A} \sim \mathrm{i}\,\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi} \equiv -s + \mathrm{i}c \quad (\xi \uparrow \infty), \tag{4.9}$$

where

$$s = \sin\tfrac{1}{2}\phi, \quad c = \cos\tfrac{1}{2}\phi. \tag{4.10}$$

5. Stationary solutions for $D > 0$

Proceeding as in II §5, we now assume $D > 0$, introduce the amplitude A, the phase ψ, the logarithmic derivative L (the Lagrangian makes no further appearance) and the wavenumber (for the envelope) K through the transformation

$$\mathscr{A} = A(\xi)\,\mathrm{e}^{\mathrm{i}\psi(\xi)}, \quad \frac{\mathrm{d}\log\mathscr{A}}{\mathrm{d}\xi} = \frac{A'}{A} + \mathrm{i}\psi' \equiv L + \mathrm{i}K, \tag{5.1a, b}$$

regard L and K as functions of

$$Z = (A/A_*)^2, \quad A_*^2 = \tfrac{1}{4}(5 - 3\cos\phi) = 2s^2 + \tfrac{1}{2}c^2, \tag{5.2a, b}$$

and transform (4.7)–(4.9) to

$$\frac{\mathrm{d}}{\mathrm{d}Z}\{Z[L^2 - s^2(1-Z)]\} = K^2 - c^2(1 + \tfrac{1}{2}Z), \quad L\frac{\mathrm{d}(ZK)}{\mathrm{d}Z} = -sc, \tag{5.3a, b}$$

$$L = -s, \quad K = c \quad (Z = 0), \tag{5.4a, b}$$

and

$$M \equiv A_*[Z(L^2+K^2)]^{\frac{1}{2}} = \sigma, \quad \psi_0 = \tan^{-1}(L/K) \quad (Z = Z_0), \tag{5.5a, b}$$

where the subscript zero implies $\xi = 0$. The integration of (5.3) may be started at the singular point $Z = 0$ and continued to that point, if any, at which $M = \sigma$ (M *vs.* Z

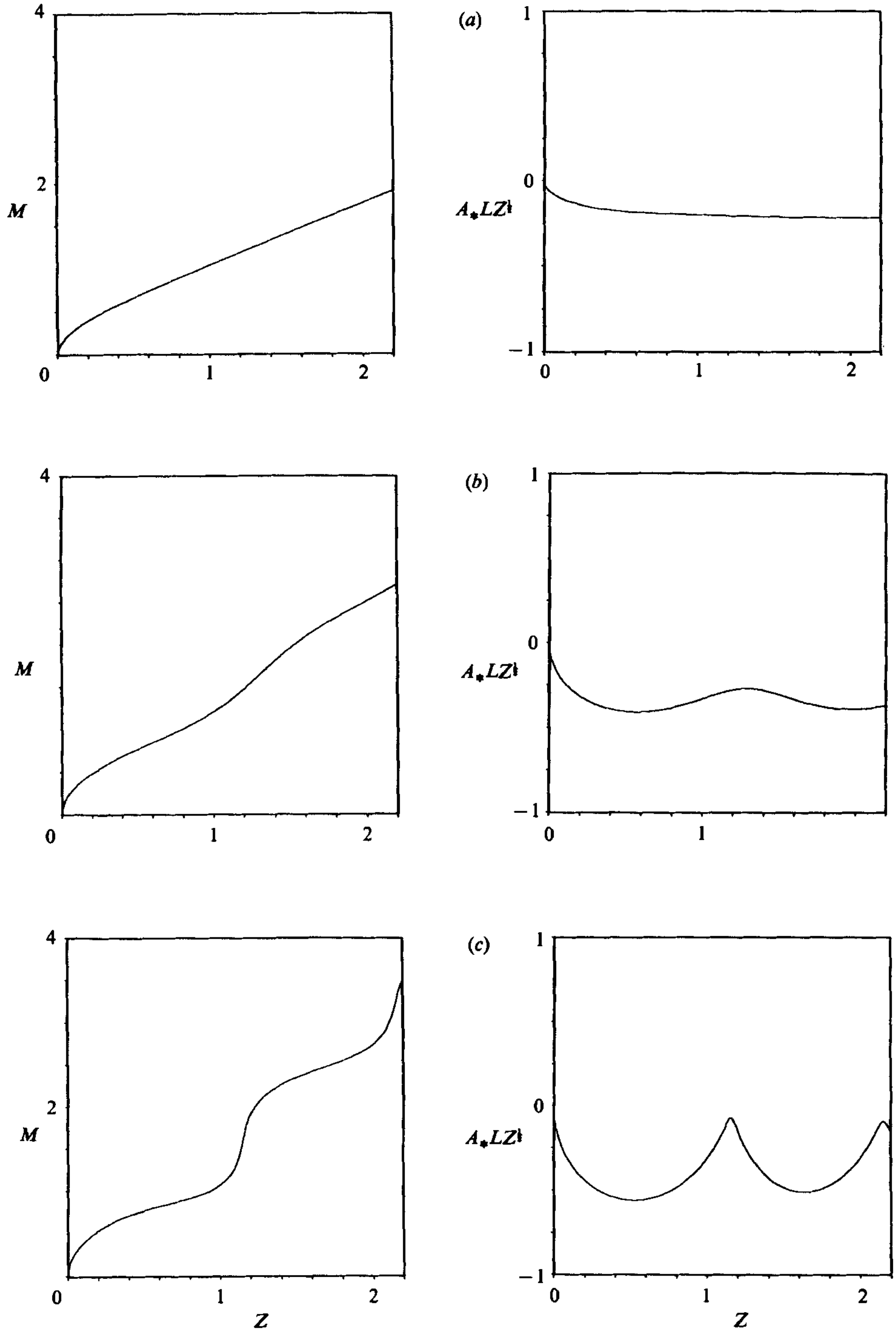

FIGURE 1 (a–c). For caption see facing page.

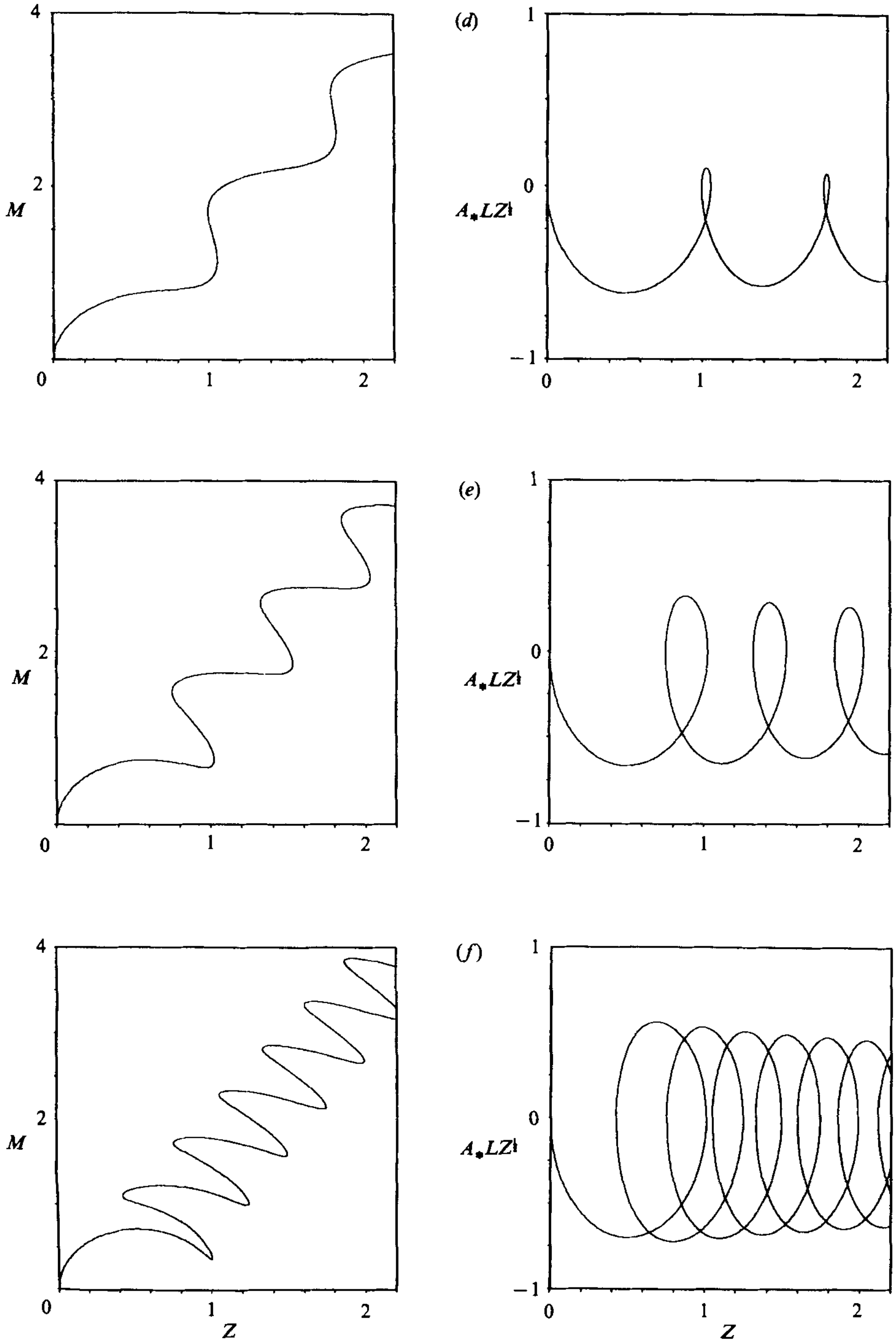

FIGURE 1. M *vs.* Z (σ *vs.* Z_0) and $A_* L Z^{\frac{1}{2}}$ *vs.* Z ($\sigma \sin \psi_0$ *vs.* Z_0) for: (*a*) $\phi = \frac{1}{4}\pi$, (*b*) $\phi = \frac{1}{2}\pi$, (*c*) $\phi = \frac{2}{3}\pi$, (*d*) $\phi = \frac{3}{4}\pi$, (*e*) $\phi = \frac{5}{6}\pi$ and (*f*) $\phi = \frac{11}{12}\pi$.

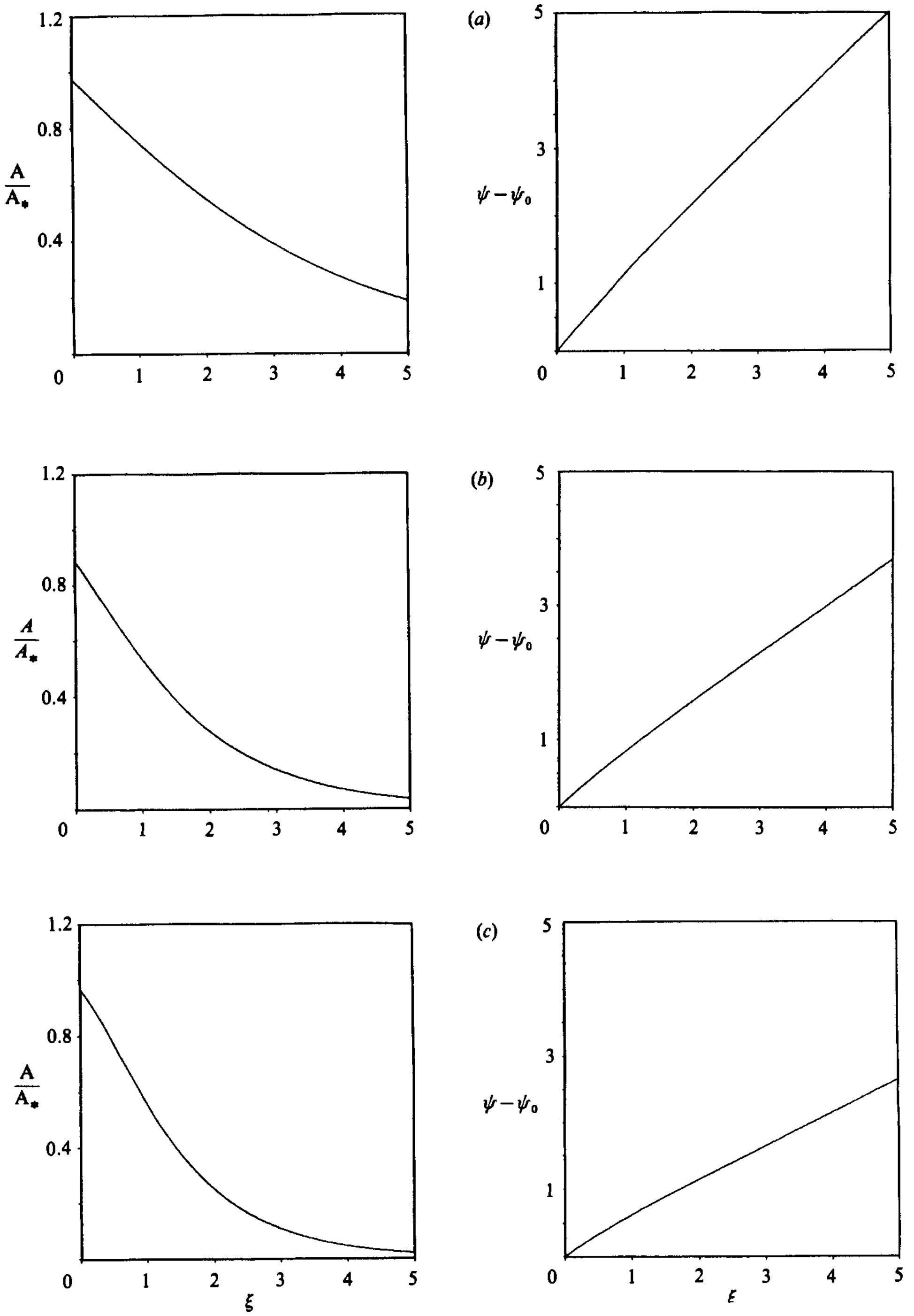

FIGURE 2(a–c). For caption see facing page.

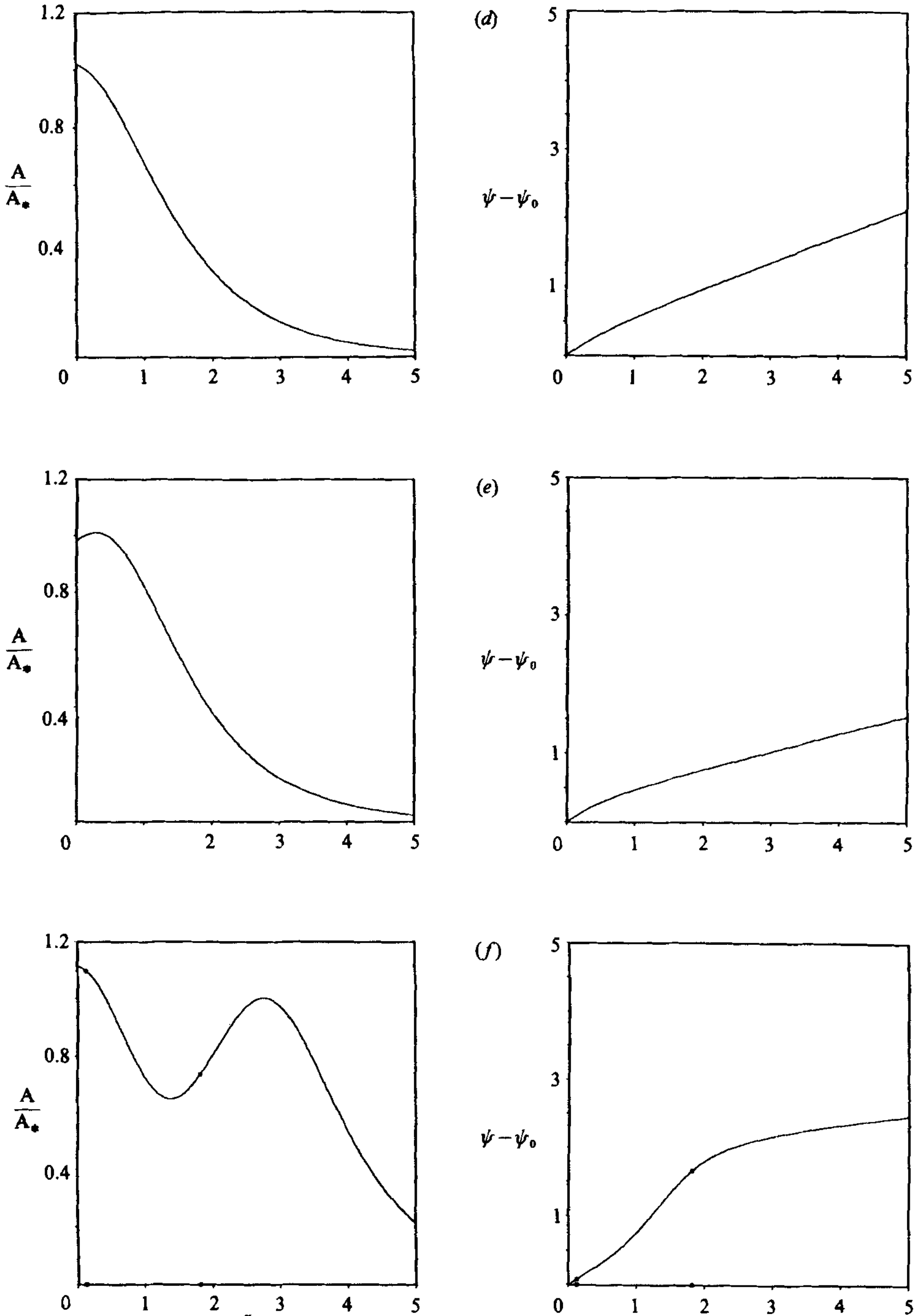

FIGURE 2. A/A_* and $\psi-\psi_0$ *vs.* ξ for $\sigma = 1$ and (*a*) $\phi = \frac{1}{4}\pi$, (*b*) $\phi = \frac{1}{2}\pi$, (*c*) $\phi = \frac{2}{3}\pi$, (*d*) $\phi = \frac{3}{4}\pi$, (*e*) $\phi = \frac{5}{8}\pi$ and (*f*) $\phi = \frac{11}{12}\pi$. Three solutions are possible for $\phi = \frac{11}{12}\pi$; the complete profiles correspond to the largest value of Z_0, while the positions of the wavemaker for the alternative solutions are indicated by the small circles.

may be interpreted as a plot of σ *vs.* Z_0 with ϕ as the family parameter, see figure 1). This determines Z_0 and ψ_0 (see figure 1) through (5.5), after which (5.1*b*) may be integrated to obtain

$$\xi = \frac{1}{2}\int_{Z_0}^{Z}\frac{\mathrm{d}Z}{ZL}, \quad \psi = \psi_0 + \frac{1}{2}\int_{Z_0}^{Z}\frac{K\mathrm{d}Z}{ZL}. \tag{5.6a, b}$$

The numerical integration of (5.3)–(5.6) is straightforward, although branch points and multiple solutions must be anticipated (cf. II §6). Analytical approximations may be obtained as in II §6. In particular, the approximations

$$L = -s(1-Z)^{\frac{1}{2}}, \quad K = 2c[1+(1-Z)^{\frac{1}{2}}]^{-1}, \tag{5.7a, b}$$

yield

$$A_*^2 Z_0\{s^2(1-Z_0)+4c^2[1\pm(1-Z_0)^{\frac{1}{2}}]^{-2}\} = \sigma^2, \tag{5.8a}$$

and

$$\tan\psi_0 = \mp\tfrac{1}{2}(1-Z_0)^{\frac{1}{2}}[1\pm(1-Z_0)^{\frac{1}{2}}]\tan\tfrac{1}{2}\phi \; (-\tfrac{1}{2}\pi < \psi_0 < \tfrac{1}{2}\pi), \tag{5.8b}$$

for the determination of Z_0 and ψ_0,

$$A = A_* \operatorname{sech}(s\xi \pm \operatorname{sech}^{-1} Z_0) = \frac{A_0}{\cosh s\xi \pm (1-Z_0)^{\frac{1}{2}}\sinh s\xi}, \tag{5.9a}$$

and

$$\psi = \psi_0 + c\left\{\xi + \left[\frac{1-(1-Z_0)^{\frac{1}{2}}}{1+(1-Z_0)^{\frac{1}{2}}}\right]^{\pm 1}\left(\frac{1-\mathrm{e}^{-2s\xi}}{2s}\right)\right\}, \tag{5.9b}$$

where the alternative signs are vertically ordered. The upper choice yields a monotonically decaying (in ξ) envelope for which the branch point of L lies outside of the domain of integration for (5.6). The lower choice corresponds to a solution for which the integration is continued through the branch point, at which L changes sign and $A(\xi)$ has a maximum. These approximations are at least qualitatively valid for $\sigma < 1$ and $0 \leqslant \phi < \frac{3}{4}\pi$ (cf. II §5), but they fail to model the n-valued ($n > 2$) solutions that are obtained by numerical integration for $\phi > \phi_* \doteqdot 127°$ (see figure 2*f*).

Letting $Z_0 \downarrow 0$ in (5.8) and (5.9) with the upper choice of signs, we obtain $A_0 \to \sigma$, $\psi_0 \to -\frac{1}{2}\phi$ and

$$\mathscr{A} \to \sigma \exp\{\mathrm{i}(\mathrm{e}^{\frac{1}{2}\mathrm{i}\phi}\xi - \tfrac{1}{2}\phi)\}, \tag{5.10}$$

which, through (4.4), (4.5), (3.1)–(3.5) and (1.4), may be shown to be equivalent to (1.7).

Equations (5.3), (5.4) and (5.6) are identical to II (6.5), (6.7) and (6.8), and the two problems differ only in the boundary conditions at the wavemaker, (5.5) *vs.* II (6.6). Results for σ *vs.* Z_0(M *vs.* Z) and $\sigma \sin\psi_0$ *vs.* Z_0($A_* LZ^{\frac{1}{2}}$ *vs.* Z), as determined by (5.5), are plotted in figure 1. The corresponding results for A and ψ for $\sigma = 1$ are plotted in figure 2.

6. Stationary solutions for $D < 0$

If $D < 0$ the transformation (5.1) carries (4.7) over to

$$\frac{\mathrm{d}}{\mathrm{d}Z}\{Z[L^2 - s^2(1+Z)]\} = K^2 - c^2(1-\tfrac{1}{2}Z), \quad L\frac{\mathrm{d}(ZK)}{\mathrm{d}Z} = -sc, \tag{6.1a, b}$$

in place of (5.3), whilst (5.4)–(5.6) are unchanged.

Proceeding as in II §6, we remark that exact solutions of (6.1) and (5.4)–(5.6) are possible if either $\phi = 0 (\alpha = 0, \beta > 0)$ and $0 < \sigma < \frac{1}{2}$ or $\phi = \pi (\alpha = 0, \beta < 0)$ and $\sigma > 0$. In the former case

$$L = 0, \quad K = (1-\tfrac{1}{2}Z)^{\frac{1}{2}}, \quad Z_0 = 1-(1-4\sigma^2)^{\frac{1}{2}}, \quad \psi_0 = 0, \tag{6.2}$$

$$A = A_0 = (\tfrac{1}{2}Z_0)^{\frac{1}{2}}, \quad \psi = \psi_0 + (1-\tfrac{1}{2}Z_0)^{\frac{1}{2}}\xi \quad (\phi = 0,\ \sigma < \tfrac{1}{2}), \tag{6.3}$$

which describes a progressive wave of constant amplitude.† In the latter case,

$$L = -(1+Z)^{\frac{1}{2}}, \quad K = 0, \quad Z_0 = \tfrac{1}{2}[(1+2\sigma^2)^{\frac{1}{2}}-1], \quad \psi_0 = -\tfrac{1}{2}\pi, \tag{6.4}$$

$$A = 2^{\frac{1}{2}} \operatorname{csch}(\xi + \sinh^{-1} Z_0^{-\frac{1}{2}}), \quad \psi = \psi_0 \quad (\phi = \pi,\ \sigma > 0), \tag{6.5}$$

which describes a trapped solitary wave.

The counterparts of (5.7)–(5.9) are (cf. (6.4) and (6.5))

$$L = -s(1+Z)^{\frac{1}{2}}, \quad K = 2c[1+(1+Z)^{\frac{1}{2}}]^{-1}, \tag{6.6a, b}$$

$$A_*^2 Z_0\{s^2(1+Z_0)+4c^2[1+(1+Z_0)^{\frac{1}{2}}]^{-2}\} = \sigma^2, \tag{6.7a}$$

$$\tan\psi_0 = -\tfrac{1}{2}(1+Z_0)^{\frac{1}{2}}[1+(1+Z_0)^{\frac{1}{2}}]\tan\tfrac{1}{2}\phi, \tag{6.7b}$$

$$A = A_* \operatorname{csch}(s\xi + \sinh^{-1} Z_0^{-\frac{1}{2}}) = \frac{A_0}{\cosh s\xi + (1+Z_0)^{\frac{1}{2}}\sinh s\xi}, \tag{6.8a}$$

$$\psi = \psi_0 + c\left\{\xi - \left[\frac{(Z_0+1)^{\frac{1}{2}}-1}{(Z_0+1)^{\frac{1}{2}}+1}\right]\left(\frac{1-\mathrm{e}^{-2s\xi}}{2s}\right)\right\}. \tag{6.8b}$$

The left-hand side of (6.7 a) is a monotonically increasing function of Z_0, by virtue of which (6.7 a) has only one root, and A decays monotonically in ξ. Comparison with the results of the integration of (6.1) reveals that (6.7) and (6.8) are in error by less than 10% for $0 \leqslant Z_0 \leqslant Z$. The graphs of A/A_* and $\psi - \psi_0$ are qualitatively similar to those in figure 2(a).

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Appendix. Comparison with Barnard, Mahony & Pritchard (1977)

BMP refer length and time to b and $(b/g)^{\frac{1}{2}}$ and refer the complex amplitude of the dominant mode to $\exp(\mathrm{i}\omega t)$ (*vs.* $\exp(-\mathrm{i}\omega t)$ in the present formulation). A comparison of the two formulations leads to the following equivalences (BMP → Miles)

$$\theta \to \epsilon, \quad \epsilon \to C(T/\pi)^{\frac{1}{2}}\epsilon, \tag{A 1a, b}$$

$$X \to \pi^{-\frac{5}{4}}T^{-\frac{3}{4}}C^{\frac{1}{2}}X, \quad A_0 \to \pi^{-\frac{5}{4}}T^{-\frac{3}{4}}C^{-\frac{1}{2}}(q+\mathrm{i}p), \tag{A 2a, b}$$

$$p+\mathrm{i}q \to \tfrac{1}{2}\pi^{\frac{7}{2}}T^{-\frac{1}{2}}(\beta-\mathrm{i}\alpha), \quad r \to \frac{\pi^6}{16}TCD, \quad \left|\frac{r}{p^2+q^2}\right|^{\frac{1}{2}} \to \sigma. \tag{A 3a, b, c}$$

† A second such solution for $\phi = 0$ may be obtained by changing the sign of the radical in (6.2), but this gives $Z_0 = 2$ (rather than $Z_0 = 0$) for $\sigma \downarrow 0$, which suggests that this second solution is unstable.

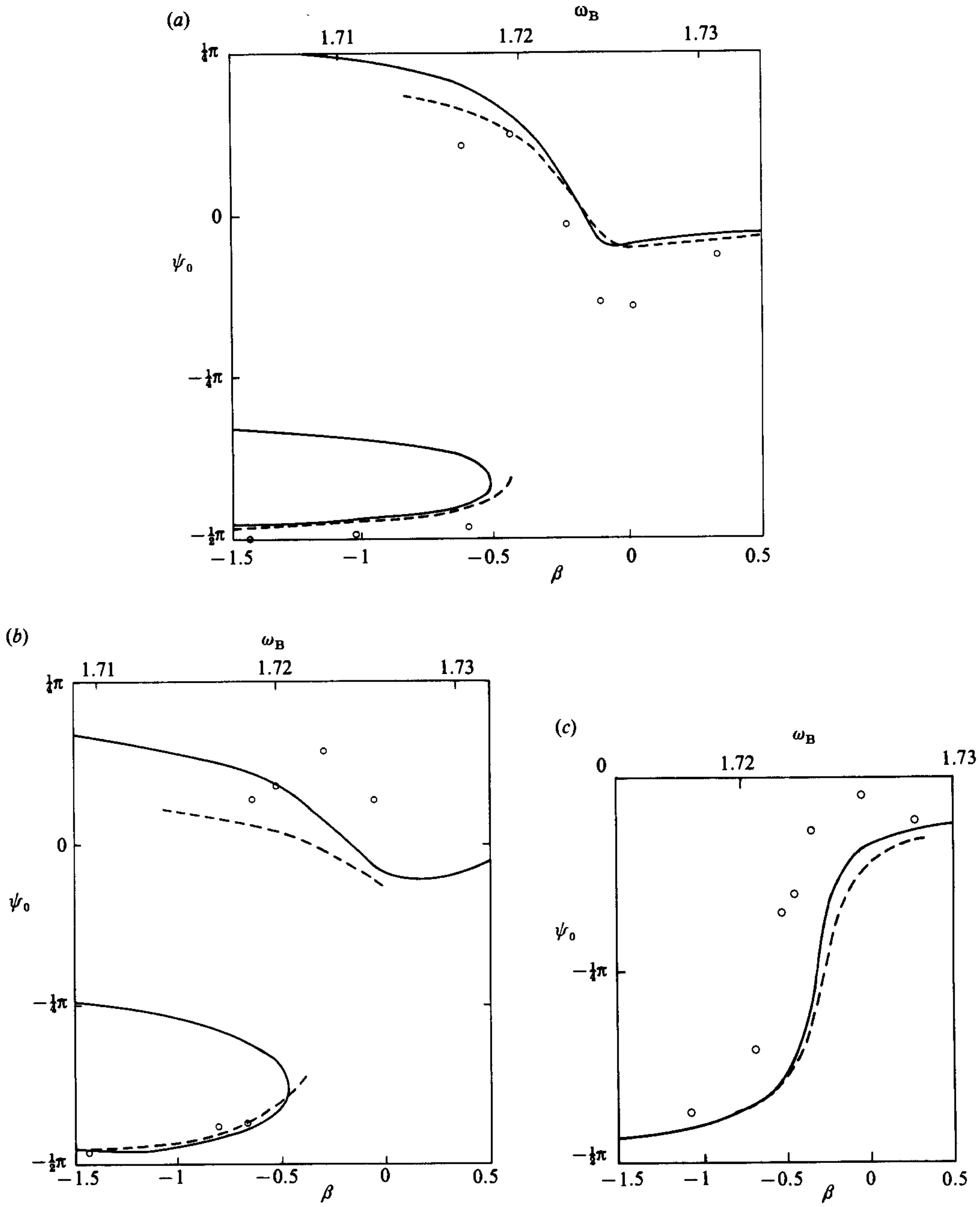

FIGURE 3. ψ_0 vs. β for $\epsilon\alpha = 0.00139$ and (a) $\epsilon = 0.00891$, (b) $\epsilon = 0.00668$, (c) $\epsilon = 0.00449$, corresponding to figures 8(a), 8(b) and 8(c) respectively, in BMP (in whose notation $\phi(0) = \frac{1}{2}\pi + \psi_0$; see also (A 1)–(A 3) herein). The solid curves are calculated from the present theory, the dashed curves are BMP's calculated results, and the small circles are BMP's experimental results.

The tentative equivalences (A3) follow from a term-by-term comparison of BMP (2.22) and (4.1) above and the invocation of (A2). In fact, (A 3*a*) cannot be satisfied, since both q and α are positive, which suggests that the sign of q in BMP (2.22) must be wrong (presumably owing to an incorrect evaluation of $i^{\frac{1}{2}}$). This conclusion also follows from the impossibility of matching their solution to the damped linear solution through a limit equivalent to that of (5.10) above.

The equivalence (A 3*b*) implies, through BMP (2.16) and (2.17),

$$D' = \frac{2T^{12}+3T^8+12T^4-9}{8T^8}, \tag{A 4}$$

($4D/\pi^4 T^2$ in BMP's notation) in place of (3.9 *a*). The difference between (3.9*a*) and (A 4) appears to stem in part from an algebraic or typographic slip, such that T^2 appears in place of T, and, more fundamentally, from BMP's adoption of the self-interaction coefficient for standing waves in a closed basin (Tadjbakhsh & Keller 1960), for which the volumetric displacement $\iint \zeta \, dx \, dy \equiv m$ must vanish. This constraint need not hold for a semi-infinite channel, since a finite value of m may be compensated by an infinitesimal change in the mean elevation over the infinite free surface to ensure conservation of mass; indeed, $m \neq 0$ is necessary in order to cancel the secular (linear in t) growth of ϕ in a channel of finite depth (see remarks following (3.6) above, Larraza & Putterman 1984 and Miles 1984). The difference vanishes for deep-water waves ($T = 1$), for which $D = D' = 1$; on the other hand, $D'/D \sim 1/(kd)^4$ as $kd \downarrow 0$.

Equations (5.3) and (5.4) above are equivalent to BMP (3.20) and (3.21) after invoking (A 2) and (A 3) and changing the sign of q, which implies a change in the sign of the phase ψ; however, BMP (3.20) and (3.21) contain three (subsequently reduced to two) parameters, whereas (5.3) and (5.4) contain only the single parameter ϕ. BMP use essentially the same methods of integration as those used here and in II, and they find that the equivalent of σ *vs.* Z_0 may be triple-valued in certain parametric ranges (see their figure 3); however, they do not report the possibility of n-valued results with $n > 3$ (cf. figure 1*d–f* above), presumably because they did not investigate values of (the equivalents of) ϕ near π for sufficiently large σ.

A comparison of the present results for A *vs.* ξ for $\sigma = 0.485$ and $\phi = 0.92\pi$ with BMP's calculated results (figure 9*c*) for ζ *vs.* X for $p = -17.32$, $q = 4.402$ and $r = 74.96$ yields agreement within the accuracy with which the plots can be read. (BMP's calculated results are within 10–20% of their experimental measurements for this case, for which $T = 0.948$ implies $D = 0.882$ and $D' = 0.712$.) BMP do not present calculated results for ψ *vs.* ξ but do give both calculated and measured results (their figure 8) for ψ_0 ($\phi(0) - \frac{1}{2}\pi$ in their notation), which are compared with the present theory in figure 3. The present results are somewhat closer to the observed values than are BMP's calculated values, possibly owing to the error in their calculation of D, but the differences are small. Perhaps the most interesting difference is that the present theory predicts three branches of ψ_0 for $\omega < \omega_1$ in figure 3(*a*, *b*) but that the intermediate branch is not observed, presumably because it is unstable.

REFERENCES

Aranha, J. A., Yue, D. K. P. & Mei, C. C. 1982 Nonlinear waves near a cut-off frequency in an acoustical duct – a numerical study. *J. Fluid Mech.* **121**, 465–485.

Barnard, B. J. S., Mahony, J. J. & Pritchard, W. G. 1977 The excitation of surface waves near a cut-off frequency. *Phil. Trans. R. Soc. Lond.* **286**, 87–123.

HAVELOCK, T. H. 1929 Forced surface waves on water. *Phil. Mag.* **8**(7), 569–576.

KIT, E., MILOH, T. & SHEMER, L. 1987 Experimental and theoretical investigation of nonlinear sloshing waves in a rectangular channel. *J. Fluid Mech.* **181**, 265–291.

LARRAZA, A. & PUTTERMAN, S. 1984 Theory of non-propagating surface-wave solitons. *J. Fluid Mech.* **148**, 443–449.

LUKE, J. C. 1967 A variational principle for a fluid with a free surface. *J. Fluid Mech.* **27**, 395–397.

MILES, J. W. 1967 Surface-wave damping in closed basins. *Proc. R. Soc. Lond.* A**297**, 459–475.

MILES, J. W. 1984 Parametrically excited solitary waves. *J. Fluid Mech.* **148**, 451–460; Corrigenda 1985 *J. Fluid Mech.* **154**, 535.

MILES, J. W. 1988 Parametrically excited, standing cross-waves. *J. Fluid Mech.* **186,** 119–127.

MILES, J. & BECKER, J. 1988 Parametrically excited, progressive cross-waves. *J. Fluid Mech.* **186**, 129–146.

TADJBAKHSH, I. & KELLER, J. B. 1960 Standing surface waves of finite amplitude. *J. Fluid Mech.* **8**, 442–451.